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Basisprüfung lineare Algebra - Winter 2017

1) a) Sieht man direkt, könnte man aber auch mittels Vektorprodukt berechnen:

$\alpha = 1 \qquad \beta = 1$

b) 1) Ähnlich zur Winter 2016 Prüfungsaufgabe 1 bilden wir einfach die Orthonormalbasis von A und berechnen R mit Hilfe von Gauss-Jordan

$A = Q\,R$

$$Q = \text{normiertes } A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Gauss-Jordan:

$$\left[\begin{array}{ccc|ccc} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 1 & 1 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -1 & 1 & 1 \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 1 & 1 & 1 \\ 0 & \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{3}} & 0 & 2 & 2 \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 & -2 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 1 & 1 & 1 \\ 0 & \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{3}} & 0 & 2 & 2 \\ 0 & 0 & \frac{3}{\sqrt{3}} & 0 & 0 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & 0 & 1 & 1 & 0 \\ 0 & \frac{2}{\sqrt{6}} & 0 & 0 & 2 & 0 \\ 0 & 0 & \frac{3}{\sqrt{3}} & 0 & 0 & 3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} \frac{1}{\sqrt{2}} & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{6}} & 0 & 0 & 2 & 0 \\ 0 & 0 & \frac{3}{\sqrt{3}} & 0 & 0 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 1 & 0 & 0 & \sqrt{3} \end{array}\right]$$

$$\Rightarrow \underline{\underline{A}} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}}_{\underline{\underline{Q}}} \underbrace{\begin{bmatrix} -\sqrt{2} & 0 & 0 \\ 0 & -\sqrt{6} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}}_{\underline{\underline{R}}}$$

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2) Achtung! $|\det(A)| \leftarrow$ Betrag

$$|\det(A)| = |\det(R)| = |-\sqrt{36}| = \underline{\underline{6}}$$

2. a)

EW: $\det \begin{bmatrix} 2-\lambda & 0 & -3 \\ 0 & 2-\lambda & 0 \\ -3 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda)((2-\lambda)^2 - 9)$

$$= (2-\lambda)^3 - 9(2-\lambda)$$

$$= (2-\lambda)^3 - 18 + 9\lambda$$

$$= -\lambda^3 + 6\lambda^2 - 12\lambda + 8 - 18 + 9\lambda$$

$$= -\lambda^3 + 6\lambda^2 - 3\lambda - 10 \quad | \lambda = 2$$

$$= (\lambda - 2)(-\lambda^2 + 4\lambda + 5)$$

$$= (\lambda - 2)(-\lambda + 5)(\lambda + 1)$$

$$\Rightarrow \underline{\underline{\lambda_1 = 2}} \quad \lambda_2 = \underline{\underline{5}} \quad \lambda_3 = \underline{\underline{-1}}$$

EV: $\lambda_1 = 2$:

$$\left[\begin{array}{ccc|c} 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|} -3 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_3 = 0 \quad x_2 = s \quad x_1 = 0$

$$\Rightarrow E_2 = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

$\lambda_2 = 5$

$$\left(\begin{array}{ccc|c} -3 & 0 & -3 & 0 \\ 0 & -3 & 0 & 0 \\ -3 & 0 & -3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|} -3 & 0 & -3 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_3 = s \quad x_2 = 0 \quad x_1 = -s$$

$$E_5 = \text{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$\lambda_3 = -1$

$$\left(\begin{array}{ccc|c} 3 & 0 & -3 & 0 \\ 0 & 3 & 0 & 0 \\ -3 & 0 & 3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 3 & 0 & -3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow x_3 = s \quad x_2 = 0 \quad x_1 = s$$

$$E_{-1} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

b) Symmetrische Matrix mit alles verschiedenen EW

$\rightarrow$ EV sind automatisch orthogonal $\rightarrow$ normieren.

Orthonormale Eigenbasis: $b^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $b^{(2)} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $b^{(3)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ ✓

c)

$$e^C = T e^D T^T \quad \Leftrightarrow \quad C = TDT^T$$

$$e^D := \begin{bmatrix} e^{d(1)} & 0 & 0 \\ 0 & e^{d(2)} & 0 \\ 0 & 0 & e^{d(3)} \end{bmatrix}$$

Finde also zuerst Diagonalmatrix $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$T = EV(A) = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$e^C = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} e^2 & 0 & 0 \\ 0 & e^5 & 0 \\ 0 & 0 & e^{-1} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{e^5}{\sqrt{2}} & \frac{e^{-1}}{\sqrt{2}} \\ e^2 & 0 & 0 \\ 0 & \frac{e^5}{\sqrt{2}} & \frac{e^{-1}}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \underline{\underline{\begin{bmatrix} \frac{e^5+e^{-1}}{2} & 0 & \frac{-e^5+e^{-1}}{2} \\ 0 & e^2 & 0 \\ \frac{-e^5+e^{-1}}{2} & 0 & \frac{e^5+e^{-1}}{2} \end{bmatrix}}}$$

3) a) 2 Kriterien: $\forall a, b \in \mathcal{P}_3$, $\forall \alpha \in \mathbb{R}$

1) $F(a+b) = F(a) + F(b)$

2) $F(\alpha a) = \alpha F(a)$

$$F(f(x)) = x f'(x) + \frac{1}{x} \int_0^x f(s)\,ds$$

Die Abbildung ist zuerst einmal wohldefiniert, da

$F(1) = 1 \in \mathcal{P}_3$, $F(x) = \frac{3}{2}x \in \mathcal{P}_3$, $F(x^2) = \frac{7}{3}x^2 \in \mathcal{P}_3$

1) & 2) kombiniert: Spart Zeit

$$F(a(x) + \alpha b(x)) = x(a(x) + \alpha b(x))' + \frac{1}{x}\left(\int_0^x (a(s) + \alpha b(s))\,ds\right)$$

$$= x a'(x) + \alpha x b'(x) + \frac{1}{x}\int_0^x a(s)\,ds + \frac{1}{x}\int_0^x \alpha b(s)\,ds$$

$$= x a'(x) + \frac{1}{x}\int_0^x a(s)\,ds + \alpha\left(x b'(x) + \frac{1}{x}\int_0^x b(s)\,ds\right)$$

$$= F(a(x)) + F(b(x))$$

b) $1 = 1 \qquad = 1 + 0x + 0x^2$

$x = \frac{3}{2}x \qquad = 0 + \frac{3}{2}x + 0x^2$

$x^2 = 2x^2 + \frac{1}{3}x^2 = \frac{7}{3}x^2 \qquad = 0 + 0x + \frac{7}{3}x^2$

$$\underline{\underline{A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{7}{3} \end{bmatrix}}}$$

wenn der Koordinaten-vektor wie folgt ist $\begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$

$$1 - x = 1 - \frac{3}{2}x = \frac{1}{2}\cdot 2 - 6\,\frac{x}{4} + 0\,\frac{x^2}{3}$$

$$2x = 3x = 0\cdot 2 + 12\,\frac{x}{4} + 0\,\frac{x^2}{3}$$

$$4x^2 = \frac{28}{3}x^2 = 0\cdot 2 + 0\,\frac{x}{4} + 28\,\frac{x^2}{3}$$

$$B = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -6 & 12 & 0 \\ 0 & 0 & 28 \end{bmatrix}$$

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4) a) SVD:

$$A^T A = \frac{1}{4}\begin{bmatrix} 2\sqrt{2} & 3 & -3 \\ 2\sqrt{2} & -3 & 3 \end{bmatrix}\begin{bmatrix} 2\sqrt{2} & 2\sqrt{2} \\ 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 26 & -10 \\ -10 & 26 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 13 & -5 \\ -5 & 13 \end{bmatrix}$$

EW: $$\det \frac{1}{2}\begin{bmatrix} 13-2\lambda & -5 \\ -5 & 13-2\lambda \end{bmatrix} = 0$$

Achtung: lässt man das $\frac{1}{2}$ ausgeklammert, so auch beim λ auskl.!

$$0 = \frac{1}{4}\left((13-2\lambda)^2 - 25\right) = (4\lambda^2 - 52\lambda + 169 - 25)$$

$$= (\lambda^2 - 13\lambda + 36)$$

$$= (\lambda - 4)(\lambda - 9)$$

$$\Rightarrow \lambda_1 = 4 \qquad \lambda_2 = 9$$

$$\underline{\sigma_1 = 3} \qquad \underline{\sigma_2 = 2}$$

EV: $\lambda_1 = 4$

$$\left[\begin{array}{cc|c} \frac{5}{2} & -\frac{5}{2} & 0 \\ -\frac{5}{2} & \frac{5}{2} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_2 = s \quad x_1 = s$$

$$\Rightarrow v^{(2)'} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \underline{v^{(2)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$\lambda_2 = 9$

$$\left[\begin{array}{cc|c} -\frac{5}{2} & -\frac{5}{2} & 0 \\ -\frac{5}{2} & -\frac{5}{2} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_2 = s \quad x_1 = -s$$

$$\Rightarrow v^{(1)'} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \underline{v^{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

$$\Rightarrow \underline{\underline{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}} \qquad \underline{\underline{S} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}}$$

U aus V & S berechnen:

$$u^{(1)} = \frac{A v^{(1)}}{s^{(1)}} = \frac{\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}}{3} = \frac{\begin{bmatrix} 0 \\ -\frac{3}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \end{bmatrix}}{3} = \underline{\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}$$

$$u^{(2)} = \frac{A v^{(2)}}{s^{(2)}} = \frac{\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}}{2} = \frac{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}}{2} = \underline{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

$$u^{(3)} = u^{(1)} \times u^{(2)} = \underline{\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} \Rightarrow \underline{\underline{U} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}}$$

$$\Rightarrow \quad S V^T c = U^T b \qquad U^T b = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$S V^T c = d$$

$$\hat{S} V^T c = d_0$$

$$S = \begin{bmatrix} \hat{S} \\ \hline 0 \end{bmatrix} \qquad d = \begin{bmatrix} d_0 \\ \hline d_1 \end{bmatrix}$$

$$\Rightarrow \quad c = V \hat{S}^{-1} d_0$$

$$\hat{S} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \qquad \det(\hat{S}) = 6$$

$$\hat{S}^{-1} = \frac{1}{6} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$c = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \sqrt{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \underline{\underline{\frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underline{x}}}$$

b) Normalengleichung: $A^T A \underline{x} = A^T b$

$$A^T A = \frac{1}{2} \begin{bmatrix} 13 & -5 \\ -5 & 13 \end{bmatrix} \qquad A^T b = \begin{bmatrix} \sqrt{2} & \frac{3}{2} & -\frac{3}{2} \\ \sqrt{2} & -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Gauss:
$$\left[\begin{array}{cc|c} 13 & -5 & 4 \\ -5 & 13 & 4 \end{array}\right] \to \left[\begin{array}{cc|c} 13 & -5 & 4 \\ 0 & 13 - \frac{25}{13} & 4 + \frac{20}{13} \end{array}\right]$$

$$\left(13 - \frac{25}{13}\right) x_2 = 4 + \frac{20}{13}$$

$$(169 - 25)\, x_2 = 52 + 20$$

$$144 \, x_2 = 72$$

$$\underline{\underline{x_2 = \frac{1}{2}}}$$

$$13x_1 - \frac{5}{2} = 4$$
$$26x_1 - 5 = 8$$
$$26x_1 = 13$$
$$\underline{\underline{x_1 = \frac{1}{2}}}$$

5. a) Kriterien: $\forall x, y \in L^2[-1,1]$:

(I) Linear im 2. Argument:

$$\langle x, ay + bz\rangle = a\langle x, y\rangle + b\langle x, z\rangle$$

(II) $\langle x, y\rangle = \langle y, x\rangle$

(III) Positiv definit:

$$\langle x, x\rangle \geq 0, \quad \langle x, x\rangle = 0 \iff x = 0$$

(I) $\langle x(t), ay(t) + bz(t)\rangle =$

$$\int_{-1}^{1} x(t)(ay(t) + bz(t))\,dt$$

$$= \int_{-1}^{1} ax(t)y(t) + bx(t)z(t)\,dt$$

$$= a\int_{-1}^{1} x(t)y(t)\,dt + b\int_{-1}^{1} x(t)z(t)\,dt = a\langle x(t), y(t)\rangle + b\langle x(t), z(t)\rangle$$ ✓

II $\langle x(t), y(t)\rangle = \int_{-1}^{1} x(t)y(t)\,dt = \int_{-1}^{1} y(t)x(t)\,dt = \langle y(t), x(t)\rangle$ ✓

III $\langle x(t), x(t)\rangle = \int_{-1}^{1} \underbrace{x(t)^2}_{\geq 0}\,dt = \frac{1}{3}x(t)^3\Big|_{-1}^{1} = \underbrace{\frac{2}{3}x(t)^3\Big|_{t=1}}_{\geq 0}$

↳ immer gerade Potenz $\quad = 0 \iff x(t) = 0$ ✓

b) Induktion: (kann argumentieren ob $0 \in \mathbb{N}$ ist oder nicht)

(i) $k=1$ & $k=2$:

$$\|f_1(t)\| = \sqrt{\langle f_1, f_1 \rangle} = \sqrt{\int_{-1}^{1} f_1(t)^2 \, dt}$$

$$= \sqrt{\int_{-1}^{1} \sin(\pi t)^2 \, dt} = \sqrt{\int_{-1}^{1} \frac{1}{2} (\underbrace{\cos(0)}_{1} - \cos(2\pi t))}$$

$$= \sqrt{\int_{-1}^{1} \frac{1}{2} dt - \frac{1}{2}\int_{-1}^{1} \cos(2\pi t) dt}$$

$$= \sqrt{1 - \underbrace{\frac{1}{2}\left(\frac{1}{2\pi} \sin(2\pi t) \Big|_{-1}^{1}\right)}_{0}} = \sqrt{1} = \underline{\underline{1}}$$

$$\|f_2(t)\| = \sqrt{\langle f_2, f_2 \rangle} = \sqrt{\int_{-1}^{1} f_2(t)^2 dt} = \sqrt{\int_{-1}^{1} \sin^2(2\pi t)}$$

$$= \sqrt{\frac{1}{2}\int_{-1}^{1} \underbrace{\cos(0)}_{1} - \cos(4\pi t) dt}$$

$$= \sqrt{1 - \underbrace{\frac{1}{2}\int_{-1}^{1} \frac{1}{4\pi} \sin(4\pi t) dt}_{0}} = \underline{\underline{1}}$$

$$\langle f_1, f_2 \rangle = \int_{-1}^{1} f_1(t) f_2(t) dt = \int_{-1}^{1} \sin(\pi t) \sin(2\pi t) dt$$

$$= \frac{1}{2}\int_{-1}^{1} \cos(-\pi t) - \cos(3\pi t) dt$$

$$= \frac{1}{2}\int_{-1}^{1} \cos(-\pi t) dt - \frac{1}{2}\int_{-1}^{1} \cos(3\pi t) dt$$

$$= -\frac{1}{2\pi} \sin(-\pi t) \Big|_{-1}^{1} - \frac{1}{6\pi} \sin(3\pi t) \Big|_{-1}^{1}$$

$$= 0 - 0 = \underline{\underline{0}}$$

✓

(ii) $n \to n+1$

$$\|f_{n+1}\| = \sqrt{\langle f_{n+1}, f_{n+1}\rangle} = \sqrt{\int_{-1}^{1} \sin^2(n\pi t + \pi t)\,dt}$$

$$= \sqrt{\frac{1}{2}\int_{-1}^{1} \underbrace{\cos(0)}_{1} - \cos(2n\pi t + 2\pi t)\,dt}$$

$$= \sqrt{1 - \frac{1}{2}\int_{-1}^{1} \cos((n+1)2\pi t)\,dt}$$

$$= \sqrt{1 - \frac{1}{2}\,\frac{1}{(n+1)2\pi}\,\underbrace{\sin((n+1)2\pi t)\,dt}_{0}} = \sqrt{1} = \underline{\underline{1}}$$

$$\langle f_n, f_{n+1}\rangle = \int_{-1}^{1} \sin(n\pi t)\sin(n\pi t + \pi t)\,dt$$

$$= \frac{1}{2}\int_{-1}^{1} \cos(2n\pi t + \pi t) - \cos(\pi t)\,dt$$

$$= \underbrace{\frac{1}{2}\,\frac{1}{2n\pi + \pi}\,\sin(\underbrace{2n\pi t + \pi t}_{(2n+1)\pi t})\Big|_{-1}^{1}}_{0} - \underbrace{\frac{1}{2\pi}\sin(\pi t)\Big|_{-1}^{1}}_{0}$$

$$= \underline{\underline{0}}$$

✓

c) Haben in b) gerade gezeigt, dass alle Vektoren $f_k : k \in \mathbb{N}$ orthonormal zueinander stehen $\Rightarrow$ sie sind linear unabhängig.

Unsicher, ob diese Argumentation an der Prüfung ausreicht?

Trotzdem Beweis:

Müssen zeigen, dass

$$\sum_{k=1}^{n} f_k x_k = 0 \iff x = 0$$

$$f_1 x_1 + \dots + f_k x_k + \dots + f_n x_n = 0$$

$\langle f_k, \cdot\rangle |$

$\langle f_k, f_1x_1 + \dots + f_kx_k + \dots + f_nx_n \rangle = 0$

$\downarrow$ linear im 2. Glied

$$\underbrace{\langle f_k, f_1\rangle}_{0} x_1 + \underbrace{\dots}_{0} + \underbrace{\langle f_k, f_k\rangle}_{1} x_k + \dots + \underbrace{\langle f_k, f_n\rangle}_{0} x_n = 0$$

$$x_k = 0$$

Multipliziere konsekutiv von links skalar mit f_i, $i \in [1, \dots, n]$

$\Rightarrow x_1, \dots, x_k, \dots, x_n = 0 \Leftrightarrow$ alle Vektoren linear unabhängig.

d) Der Vektorraum ist unendlichdimensional, wir können alleine schon unendlich viele lin. unabh. Vektoren wie in dieser Aufgabe konstruieren.

6. a) false (vgl. in MatLab)

b) true HLGS hat nicht triviale Lösung $\Leftrightarrow$ LGS hat entweder keine oder ∞ Lösungen

c) true $\Rightarrow$ A ist diagonalisierbar, A und D sind ähnlich $\Rightarrow \det(A) = \det(D) = \lambda_1 \cdot \ldots \cdot \lambda_n \neq 0$

d) false $\Rightarrow |\det(A)| = 1 \Rightarrow \det(A) = \pm 1$

e) <u>false</u> hat ein Quadrat drin

$\Rightarrow f(a+b) \neq f(a) + f(b)$

f) $A^T A = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

EW: $\lambda_1 = 3 \quad \lambda_2 = 2$

$\sigma_1 = \sqrt{3} \quad \sigma_2 = \sqrt{2}$

$\Rightarrow$ <u>false</u>